SYDNEY GRAMMAR SCHOOL — TRIAL EXAMINATION 1999

3/4 UNIT MATHEMATICS FORM VI

Time allowed: 2 hours (plus 5 minutes reading) **Exam date:** 16th August, 1999

Instructions:

All questions may be attempted.
All questions are of equal value.
Part marks are shown in boxes in the left margin.
All necessary working must be shown.
Marks may not be awarded for careless or badly arranged work.
Approved calculators and templates may be used.
A list of standard integrals is provided at the end of the examination paper.

Collection:

Each question will be collected separately.
Start each question in a new answer booklet.
If you use a second booklet for a question, place it inside the first. Don't staple.
Write your candidate number on each answer booklet.

QUESTION ONE (Start a new answer booklet)

Marks

2 (a) Find and simplify the term in x^5 in the expansion of $(2-x)^7$.

2 (b) Differentiate $e^{2x}\sin x$.

2 (c) Find the gradient of the tangent to $y=\sin^{-1}\dfrac{x}{2}$ at the point where $x=1$.

2 (d) Solve $x^2-x-6>0$.

2 (e) Find, correct to the nearest minute, the acute angle between the lines $x-y+3=0$ and $2x+y+1=0$.

2 (f) Find:

(i) $\displaystyle\int \frac{1+e^x}{e^x}\,dx$,

(ii) $\displaystyle\int \frac{e^x}{1+e^x}\,dx$.

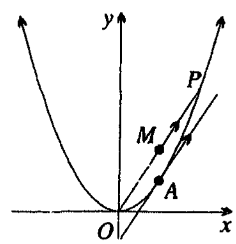
QUESTION TWO (Start a new answer booklet)

Marks

$\boxed{2}$ (a) Find the general solution of $\cos x = -\frac{1}{2}$.

$\boxed{2}$ (b) What are the coordinates of the focus of the parabola $(x+3)^2 = 8(y-1)$?

$\boxed{4}$ (c)



The point $P(2ap, ap^2)$ and the origin O lie on the parabola $x^2 = 4ay$. M is the mid-point of the chord OP.

(i) Find the gradient of OP.

(ii) Show that the tangent at a point $T(2at, at^2)$ on the parabola has gradient t.

(iii) Hence find the point A on the parabola where the tangent is parallel with the chord OP, and show that A is equidistant from M and the x-axis.

$\boxed{4}$ (d) (i) Show $\alpha^3 + \beta^3 = (\alpha+\beta)\big((\alpha+\beta)^2 - 3\alpha\beta\big)$.

(ii) Given α and β are roots of the quadratic equation $x^2 + 3x - 2 = 0$, find the value of $\alpha^3 + \beta^3$ without finding the values of the roots.

QUESTION THREE (Start a new answer booklet)

Marks

$\boxed{4}$ (a) (i) Prove that $\sin^2\theta = \frac{1}{2} - \frac{1}{2}\cos 2\theta$.

(ii) Hence determine $\displaystyle\int_0^{\pi} \sin^2\theta \, d\theta$.

$\boxed{4}$ (b) Use the substitution $u = 1 - x$ to help evaluate $\displaystyle\int_0^1 (1+3x)(1-x)^7 dx$.

$\boxed{4}$ (c) (i) Write down a value of θ for which $\dfrac{1}{1+\sin\theta}$ is undefined.

(ii) Show that $\dfrac{1}{1+\sin\theta} = \sec^2\theta - \sec\theta\tan\theta$.

(iii) Hence find $\displaystyle\int \frac{1}{1+\sin\theta}\, d\theta$. [HINT: You may want to consult the list of standard integrals.]

QUESTION FOUR (Start a new answer booklet)

Marks

3 (a) (i) Use sigma notation to express $(1+x)^{2n}$ as a sum of powers of x.

(ii) Hence show that $\sum_{r=0}^{2n} {}^{2n}C_r \left(-\frac{1}{2}\right)^r = \left(\frac{1}{2}\right)^{2n}$.

(iii) Hence evaluate $\sum_{r=0}^{2n-1} {}^{2n}C_r \left(-\frac{1}{2}\right)^r$.

4 (b) (i) Expand $\left(x - \dfrac{1}{x}\right)^2$.

(ii) Show that $\left(x^2 + \dfrac{1}{x^2}\right)^{14} = \sum_{r=0}^{14} {}^{14}C_r x^{28-4r}$.

(iii) Hence show that the coefficient of x^6 in the expansion of $\left(x - \dfrac{1}{x}\right)^2 \left(x^2 + \dfrac{1}{x^2}\right)^{14}$ is equal to ${}^{15}C_6$.

5 (c) (i) An amount P is borrowed from a bank at an interest rate of R per month compounded monthly. At the end of each month, an instalment M is paid back to the bank. Let A_n be the amount owed at the end of the n^{th} month, after the instalment is paid. Show that:

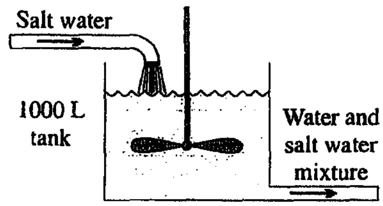
$$A_n = P(1+R)^n - \frac{M\big((1+R)^n - 1\big)}{R}.$$

(ii) A couple want to borrow \$20 000 from the bank, for a new car. After all charges are taken into account, the effective interest rate for the personal loan is 1·2% per month, compounded monthly, with the loan to be repaid over 5 years. The couple can only afford to make repayments of \$450 per month. Will the bank give them the loan? Justify your answer.

QUESTION SIX (Start a new answer booklet)

Marks

[8] (a)



In the diagram above, a tank initially contains 1000 L of pure water. Salt water begins pouring into the tank from a pipe and a stirring blade ensures it is completely mixed with the pure water. A second pipe draws the water and salt water mixture off at the same rate, so that there is always a total of 1000 L in the tank.

(i) If the salt water entering the tank contains 2 grams of salt per litre and is flowing in at the constant rate of w L/min, how much salt is entering the tank per minute?

(ii) If Q grams is the amount of salt in the tank at time t, how much salt is in 1 L at time t?

(iii) Hence write down the amount of salt leaving the tank per minute.

(iv) Use the previous parts to show that $\dfrac{dQ}{dt} = -\dfrac{w}{1000}(Q - 2000)$.

(v) Show that $Q = 2000 + Ae^{-\frac{wt}{1000}}$ is a solution of this differential equation.

(vi) Determine the value of A.

(vii) What happens to Q as $t \to \infty$?

(viii) If there is 1 kg of salt in the tank after $5\frac{3}{4}$ hours, find w.

[4] (b) A pupil investigated a differentiable function $f(x)$ and found the following information: $f(x)$ has its only zero at $x = -1$, $f(0) = 2$, $\lim\limits_{x \to \infty} f(x) = 0$.

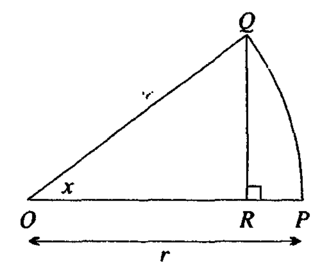
(i) Draw a graph of the possible shape of $f(x)$.

(ii) Use your graph to demonstrate that $f(x)$ must have an inflexion point to the right of $x = -1$.

QUESTION SEVEN (Start a new answer booklet)

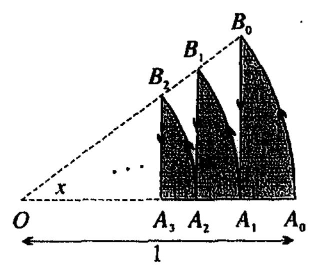
Marks

7 (a) (i) In the diagram on the right, PQ is the arc of a circle with radius r subtending an acute angle x at the centre O. R is the foot of the perpendicular from Q to the radius OP. Find lengths of the arc PQ and the interval QR in terms of x and r.



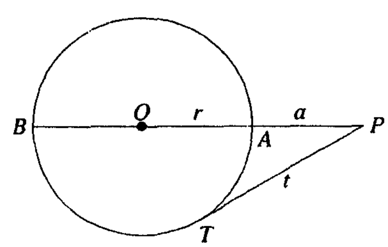
(ii) An ant travels from A_0 to O along the saw-tooth path as shown in the diagram on the right. Show that the total distance y travelled by the ant is:

$$y = \frac{x + \sin x}{1 - \cos x}.$$



(iii) Given $0 < x \le \frac{\pi}{2}$, use the derivative of y to find the value of x that gives the shortest such distance.

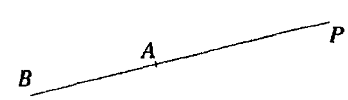
5 (b) (i) In the diagram, P is a point outside a circle with centre O and radius r. The secant PO cuts the circle at A and B respectively, and $PA = a$. PT is tangent to the circle at T and $PT = t$.



(α) Give a reason why $t^2 = a(a + 2r)$.

(β) Solve this equation for a and hence show the geometric mean of PA and PB is less than the arithmetic mean. NOTE: The geometric mean of a and b is $\sqrt{ab}$ and arithmetic mean is $\frac{a+b}{2}$.

(ii) The diagram on the right shows the interval PAB. A circle is drawn to pass through A and B. A tangent is drawn from P to touch the circle at T. Find and describe the locus of T for all such circles and tangents.



DNW

6MM1 Trial Exam Solutions

1 a) Term in $x^5 = {}^7C_5 \, 2^2 (-x)^5$ (1)

$= -84x^5$ (1)

b) $y = e^{2x} \sin x$

$\frac{dy}{dx} = 2e^{2x} \sin x + e^{2x} \cos x$ (1) + (1)

$= e^{2x}(2\sin x + \cos x)$

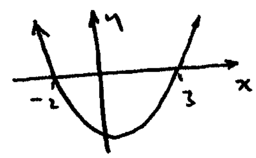
c) $\frac{dy}{dx} = \frac{1}{\sqrt{4-x^2}}$ (1)

so gradient at $x=1$ is $\frac{1}{\sqrt{4-1}} = \frac{1}{\sqrt{3}}$ (1)

d) $(x-3)(x+2) > 0$ (1)

from the graph (1)

$x < -2$ or $x > 3$



e) Let ϕ be the angle

$\tan\phi = \left| \frac{1-(-2)}{1+1\times-2} \right|$ (1)

$= 3$

so $\phi = 71°34'$ (to nearest minute) (1)

f) (i) $\int \frac{1+e^x}{e^x}\,dx = \int e^{-x} + 1 \, dx$

$= -e^{-x} + x + c$ (1)

(ii) $\int \frac{e^x}{1+e^x}\,dx = \log(1+e^x) + c.$ (1)

12

2/ a) $\cos x = -\frac{1}{2}$

so x is in 2nd or 3rd quadrants ①

thus $x = \frac{2\pi}{3} + 2n\pi$ or $-\frac{2\pi}{3} + 2n\pi$ ①

b) vertex is $(-3,1)$, focal length $=2$, axis vertical ①

so focus is $(-3,3)$ ①

c) (i) gradient $OP = \frac{ap^2}{2ap} = \frac{p}{2}$ ①

(ii) $\frac{dy}{dx} = \frac{(dy/dt)}{(dx/dt)}$

$= \frac{2at}{2a}$ ①

$= t$

(iii) thus at A parameter $t = \frac{p}{2}$

so $A = \left(ap, \frac{ap^2}{4}\right)$ ①

and $M = \left(ap, \frac{ap^2}{2}\right)$

clearly y-coord of M is twice y-coord of A, as required ①

d) (i) Expand RHS or

$(\alpha+\beta)^3 = \alpha^3 + 3\alpha^2\beta + 3\alpha\beta^2 + \beta^3$ ①

$= \alpha^3 + \beta^3 + 3\alpha\beta(\alpha+\beta)$

so $\alpha^3+\beta^3 = (\alpha+\beta)\left[(\alpha+\beta)^2 - 3\alpha\beta\right]$ ①

or ...

(ii) here $\alpha+\beta = -3$ and $\alpha\beta = -2$

so $\alpha^3+\beta^3 = -3\left[(-3)^2 - 3\times -2\right]$ ①

$= -45$ ①

12

3\. a) (i) $RHS = \frac{1}{2}(1-\cos 2\theta)$ ①

$= \frac{1}{2}(1-\cos^2\theta+\sin^2\theta)$

$= \frac{1}{2}\cdot 2\sin^2\theta$ ①

$= LHS$ #

(ii) $\int_0^{\pi}\sin^2\theta\, d\theta = \int_0^{\pi}\frac{1}{2}(1-\cos 2\theta)\, d\theta$

$= \frac{1}{2}\left[\theta - \frac{1}{2}\sin 2\theta\right]_0^{\pi}$ ①

$= \frac{\pi}{2}$ ①

b) $u = 1-x$

at $x=0$ $u=1$ and at $x=1$ $u=0$ ①

$x = 1-u$

$dx = -du$

so $\int_0^1 (1+3x)(1-x)^7\, dx = \int_1^0 (4-3u)\, u^7 . (-du)$ ①

$= \int_0^1 4u^7 - 3u^8\, du$ ①

$= \left[\frac{u^8}{2} - \frac{u^9}{3}\right]_0^1$

$= \frac{1}{6}$ ①

c) (i) when $1+\sin\theta = 0$

ie $\theta = \frac{3\pi}{2} + 2n\pi$. ①

(ii) $LHS = \frac{1}{1+\sin\theta}\cdot\frac{1-\sin\theta}{1-\sin\theta}$ ①

$= \frac{1-\sin\theta}{\cos^2\theta}$ ①

$= \sec^2\theta - \sec\theta\tan\theta$

$= RHS$ #

(iii) $\int \frac{1}{1+\sin\theta}\, d\theta = \int \sec^2\theta - \sec\theta\tan\theta\, d\theta$

$= \tan\theta - \sec\theta + c$ ①

12

4\) a) (i) $(1+x)^{2n} = \sum_{r=0}^{2n} {}^{2n}C_r \, x^r$ ①

(ii) at $x = -\frac{1}{2}$ ①

$$\left(\tfrac{1}{2}\right)^{2n} = \sum_{r=0}^{2n} {}^{2n}C_r \left(-\tfrac{1}{2}\right)^r$$

(iii) $$\left(\tfrac{1}{2}\right)^{2n} = \sum_{r=0}^{2n-1} {}^{2n}C_r \left(-\tfrac{1}{2}\right)^r + \left(\tfrac{1}{2}\right)^{2n} \quad \text{from part (ii)}$$

thus $\sum_{r=0}^{2n-1} {}^{2n}C_r \left(-\tfrac{1}{2}\right)^r = 0$ ①

b) (i) $x^2 - 2 + \frac{1}{x^2}$ ①

(ii) $\left(x^2 + \frac{1}{x^2}\right)^{14} = \sum_{r=0}^{14} {}^{14}C_r (x^2)^{14-r} (x^{-2})^r$ ①

$$= \sum_{r=0}^{14} {}^{14}C_r \, x^{28-4r}$$

(iii) $\left(x - \frac{1}{x}\right)^2 \left(x^2 + \frac{1}{x^2}\right)^{14} = \left(x^2 - 2 + \frac{1}{x^2}\right) \sum_{r=0}^{14} {}^{14}C_r \, x^{28-4r}$

$$= \sum_{r=0}^{14} {}^{14}C_r \, x^{30-4r} - 2\sum_{r=0}^{14} {}^{14}C_r \, x^{28-4r} + \sum_{r=0}^{14} {}^{14}C_r \, x^{26-4r}$$

x^6 term comes from $r = 6$ in 1st sum ①
and $r = 5$ in last sum

so coeff of $x^6 = {}^{14}C_6 + {}^{14}C_5$ ①

$= {}^{15}C_6$ by the recurrence relation (Pascal's Δ)

c) (i) $A_0 = P$

$A_1 = P(1+R) - M$ ①

$A_2 = P(1+R)^2 - M(1+R) - M$

$\vdots$

$A_n = P(1+R)^n - M\left[(1+R)^{n-1} + \ldots + (1+R) + 1\right]$ ①

$= P(1+R)^n - \dfrac{M\left[(1+R)^n - 1\right]}{R}$

(ii) Here $A_n = 0$ so $P = \dfrac{M\left[(1+R)^n - 1\right]}{R(1+R)}$ ①

and $M = 450$, $R = 0.012$, $n = 60$

for which $P \doteqdot 19168 < 20\,000$ ①

The bank will not give them the loan ①

5) a) (i) $\dot{x} = -3\sin 3t + 6\cos 3t$

$\ddot{x} = -9\cos 3t - 18\sin 3t$

$= -3^2(\cos 3t + 2\sin 3t)$ ①

and $n = 3$

(ii) $r^2 = 1^2 + 2^2 = 5$

so $\sin\alpha = \frac{1}{\sqrt{5}}$ and $\cos\alpha = \frac{2}{\sqrt{5}}$ ①

thus $r = \sqrt{5}$, $\alpha = \sin^{-1}\left(\frac{1}{\sqrt{5}}\right)$ ①

$(\doteqdot 0.46 \text{ rads})$

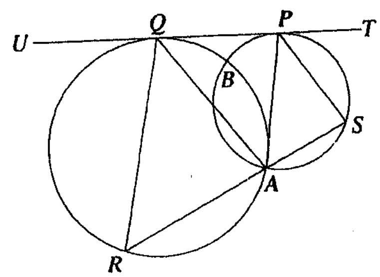
(iii) $r\sin(3t + \alpha) = 2$ ①

so $3t = \sin^{-1}\left(\frac{2}{\sqrt{5}}\right) - \sin^{-1}\left(\frac{1}{\sqrt{5}}\right)$

$t = \frac{1}{3}\left[\sin^{-1}\left(\frac{2}{\sqrt{5}}\right) - \sin^{-1}\left(\frac{1}{\sqrt{5}}\right)\right]$ ①

$\doteqdot 0.2$ to 1 dec. pl.

b)



(i) exterior angle of cyclic quadrilateral. ①

(ii) $\angle QAR = \angle UQR$ (angle in alternate segment) ①

$= \angle PSA$

hence $QA \parallel PS$ (corresponding angles equal) ①

(iii) $\angle PAS = \angle TPS$ (angle in alt segment)

$= \angle QRA$ (exterior angle of cyclic quadrilateral) ①

hence in ΔQRA and ΔPAS

$\angle QAR = \angle PSA$ proven

$\angle PAS = \angle QRA$ proven ①

thus $\Delta QRA \parallel\!\mid \Delta PAS$ (AA)

c) (i) $4a$ ①

(ii) in the right hand graph, the focal length is longer but the latus rectum is shorter. ①

12

6. a) (i) 2ω g/min (1)

(ii) $\dfrac{Q}{1000}$ g/L (1)

(iii) $\dfrac{Q\omega}{1000}$ g/min (1)

(iv) $\dfrac{dQ}{dt}$ = inflow − outflow

$= 2\omega - \dfrac{Q\omega}{1000}$ (1)

$= -\dfrac{\omega}{1000}\,(Q - 2000)$

(v) $LHS = -\dfrac{\omega}{1000} \cdot A e^{-\omega t/1000}$

$RHS = \dfrac{-\omega}{1000}\left(\cancel{2000} + A e^{-\omega t/1000} - \cancel{2000}\right)$ (1)

$= \dfrac{-\omega}{1000} A e^{-\omega t/1000}$

$=$ LHS. #

(vi) at $t = 0$ $Q = 0$ so $A = -2000$ (1)

and $Q = 2000\left(1 - e^{-\omega t/1000}\right)$

(vii) as $t \to \infty$, $e^{-\omega t/1000} \to 0$ hence $Q \to 2000$ (1)

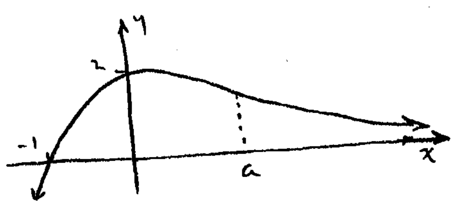
(viii) $1000 = 2000\left(1 - e^{-\omega 345/1000}\right)$

$e^{\omega 345/1000} = 2$

$\omega = \dfrac{1000}{345} \log 2$ (1)

$(\doteqdot 2$ L/min.$)$

b) (i)



(other graphs are possible) (1) + (1)

(ii) for $x < a$ f(x) is concave down (1)

for $x > a$ f(x) is concave up

Hence f(x) changes concavity and there is (1)

an inflexion point.

More precisely, for the curve to rise from & return to the x-axis it must be concave [illegible] some domain. Also f(x) must

12

7/ a) (i) $PQ = rx$

$QR = r\sin x$ (1)

(ii) along each tooth of radius r the ant travels $r(x+\sin x)$

each successive tooth has radius $\cos x$ times the previous (1)

so $y = (x+\sin x) + \cos x\,(x+\sin x) + \cos^2 x\,(x+\sin x) + \dots$ (1)

$$= \frac{x+\sin x}{1-\cos x}$$ (1)

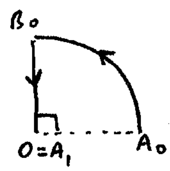
(iii)
$$y' = \frac{(1-\cos x)(1+\cos x) - (x+\sin x)(\sin x)}{(1-\cos x)^2}$$

$$= \frac{\sin^2 x - x\sin x - \sin^2 x}{(1-\cos x)^2}$$

$$= \frac{-x\sin x}{(1-\cos x)^2} < 0 \text{ for } 0 < x \leq \frac{\pi}{2}$$ (1)

ie y is decreasing so min is at right hand end pt (1)

$$y\left(\tfrac{\pi}{2}\right) = \frac{\frac{\pi}{2}+1}{1-0} = \frac{\pi}{2}+1$$ (1)



b) (i) (α) the square of the tangent is equal to the product of the intercepts of the secant (1)

(β) $a^2 + 2ar - t^2 = 0$

$a^2 + 2ar + r^2 = t^2 + r^2$

$(a+r)^2 = t^2 + r^2$

$a + r = \sqrt{t^2+r^2}$

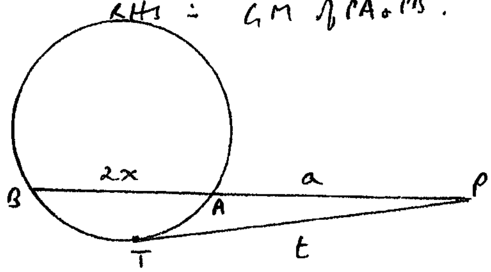
$\geq t$ (1)

with equality when $r = 0$.

LHS is AM of PA & PB

RHS is GM of PA & PB. (1)

(ii)



$t^2 = a(a+2x)$ so t is constant

so locus is the circle centre P radius t (1)